

Notes on regularity properties of infinite-dimensional Lie groups

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Abstract

The evolution of a C^k -regular Lie group G is shown to be smooth as a map to $C^{k+1}([0, 1], G)$. We also give examples of non-regular Lie groups (modelled on non-Mackey complete spaces) for which some fundamental Lie-theoretic statements are false (whose counterparts for regular Lie groups are true and well-known). Moreover, a C^1 -regular Lie group is described that is not C^0 -regular.

Introduction and statement of the results

Let G be a Lie group modelled on a locally convex space (with neutral element e and Lie algebra $L(G) := T_e(G)$), as in [5], [12], and [20] (cf. [18] for the case of sequentially complete modelling spaces). Given $g \in G$, we use the tangent map of the left translation $\lambda_g: G \rightarrow G, x \mapsto gx$ to define

$$g.v := (T_x \lambda_g)(v) \in T_{gx}(G) \quad \text{for } x \in G, v \in T_x(G).$$

Let $k \in \mathbb{N}_0 \cup \{\infty\}$. The Lie group G is called C^k -regular if the initial value problem

$$\eta'(t) = \eta(t). \gamma(t), \quad \eta(0) = e$$

has a solution $\text{Evol}(\gamma) := \eta \in C^{k+1}([0, 1], G)$ for each $\gamma \in C^k([0, 1], L(G))$, and the map

$$\text{evol}: C^k([0, 1], L(G)) \rightarrow G, \quad \gamma \mapsto \eta(1) = \text{Evol}(\gamma)(1)$$

is smooth (see [20]; cf. [10], where the property is referred to as *strong* C^k -regularity). Let $C_*^{k+1}([0, 1], G) := \{\gamma \in C^{k+1}([0, 1], G) : \gamma(0) = e\}$, endowed with the Lie group structure recalled in 1.10. Our main result is the following:

Theorem A *If G is a C^k -regular Lie group, then the map*

$$\text{Evol}: C^k([0, 1], L(G)) \rightarrow C_*^{k+1}([0, 1], G)$$

is a C^∞ -diffeomorphism. Notably, $\text{Evol}: C^k([0, 1], L(G)) \rightarrow C^{k+1}([0, 1], G)$ is smooth.

A Lie group is called *regular* if it is C^∞ -regular (see [20]; cf. [18] for the sequentially complete case, and [15] for corresponding notions in the convenient setting of analysis).¹ The special case $k = \infty$ of Theorem A (i.e., the special case of regular Lie groups) is known; see [21, Lemma A.5(1)]. We recall that the modelling space of a regular Lie group is necessarily Mackey complete (see [20, Remark II.5.3 (b)]). We refer to [14, 2.14] for the definition of Mackey completeness. Let $r \in \mathbb{N} \cup \{\infty\}$. It is known that a locally convex space E is Mackey complete if and only if the weak integral $\int_0^1 \gamma(t) dt$ exists in E for each C^r -curve $\gamma: [0, 1] \rightarrow E$ (cf. [14, 2.14]). The additive group of a locally convex space E is regular if and only if E is Mackey complete [20, Proposition II.5.6].

Let us say that a Lie group is *1-connected* if it is connected and simply connected. Consider the following statements:

- (a) If H is a 1-connected Lie group, G a Lie group and $\phi: L(H) \rightarrow L(G)$ a continuous Lie algebra homomorphism, then there is a smooth group homomorphism $\Phi: H \rightarrow G$ with $L(\Phi) = \phi$ (where $L(\Phi) := T_1(\phi)$).
- (b) If G and H are 1-connected Lie groups such that $L(G) \cong L(H)$ as a topological Lie algebra, then $G \cong H$ as a Lie group.
- (c) If G is a 1-connected abelian Lie group, then $G \cong E/\Gamma$ for the additive group $(E, +)$ of some locally convex space E and some discrete additive subgroup $\Gamma \subseteq E$.
- (d) For each $v \in L(G)$, there is a smooth homomorphism $\gamma_v: (\mathbb{R}, +) \rightarrow G$ such that $\gamma_v'(0) = v$.

It is well-known that (a) holds if G is regular, whence (b) holds if G and H are regular (see Theorem III.1.5 and Corollary III.1.6 in [20]; cf. [18] for the sequentially complete case). If G is regular, the preceding implies (c) (as in [19, Remark 3.13]; cf. [17] for an analogous result in the convenient setting). Finally, (d) is a well-known consequence of regularity of G (because $\gamma_v = \text{Evol}(t \mapsto v)$; see [20, Remark II.5.3 (a)] or [18]). As our second result, we show that all of these familiar facts become false for suitable non-regular Lie groups modelled on non-Mackey complete spaces.

¹Compare also [22] for an earlier concept of regularity in Fréchet-Lie groups, which is stronger than C^0 -regularity.

Theorem B. *All of the preceding statements (a), (b), (c) and (d) are false for suitable examples of 1-connected abelian Lie groups G and H modelled on non-Mackey complete spaces, and the counterexample for (d) can be chosen such that $L(G) \neq \{0\}$ and γ_v exists for no non-zero vector $v \in L(G)$. Moreover, there is a connected infinite-dimensional Lie group K_1 such that the only group homomorphism $(\mathbb{R}, +) \rightarrow K_1$ is the trivial map $t \mapsto e$.*

The negative answer to (b) answers a problem by Neeb [20, Problem II.3]. Whether Mackey-completeness or sequential completeness of the modelling spaces suffices for a positive answer, without the assumption of regularity (as originally asked by Milnor) remains unknown.

Let us say that a locally convex space E is *integral complete* if the weak integral $\int_0^1 \gamma(t) dt$ exists in E , for each continuous path $\gamma: [0, 1] \rightarrow E$. It is known that this property is equivalent to the metric convex compactness property, meaning that the closed convex hull of every metrizable compact subset $K \subseteq E$ is compact [24]. Varying Remark II.5.3 (b) and Proposition II.5.6 from [20], we also show:

Theorem C.

- (a) *Each C^0 -regular Lie group has an integral complete modelling space.*
- (b) *The additive group of a locally convex space E is a C^0 -regular Lie group if and only if E is integral complete.*
- (c) *The additive group of a locally convex space E is a C^1 -regular Lie group if and only if E is Mackey complete.*
- (d) *There is a Lie group which is C^1 -regular but not C^0 -regular.*

We mention that every finite-dimensional Lie group (and every Banach-Lie group) is C^0 -regular [20]. The group of compactly supported diffeomorphisms of a finite-dimensional smooth manifold M (cf. [16]) is C^0 -regular [11] (cf. [22] if M is compact). Countable direct limits of finite-dimensional Lie groups are C^1 -regular [9]. Criteria for the C^0 -regularity and C^1 -regularity of ascending unions of Banach-Lie groups can be found in [4].

1 Preliminaries and notation

We write $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{\infty\}$. All locally convex spaces are assumed Hausdorff. The terms “Lie group” and “manifold” refer to Lie groups and manifolds modelled on locally convex spaces, as in [5], [12] and [20]. The notion of C^k -map between locally convex spaces is that of Keller’s C_c^k -theory (see [5], [16], [18], [20] for expositions in varying generality), as extended in [12] to the case of maps on locally convex subsets of locally convex spaces (each point of which has a relatively open, convex neighbourhood):

1.1 Let E and F be locally convex spaces, $r \in \mathbb{N}_0 \cup \{\infty\}$ and $U \subseteq E$ be a subset. If U is open, then a map $f: U \times V \rightarrow F$ is called C^r if the iterated directional derivatives

$$d^i f(x, v_1, \dots, v_i) := (D_{v_i} \cdots D_{v_1} f)(x)$$

exist for all $i \in \mathbb{N}_0$ with $i \leq r$ and $x \in U$, $v_1, \dots, v_i \in E$, and $d^i f: U \times E^i \rightarrow F$ is continuous. If $U \subseteq E$ is a locally convex subset with dense interior U^0 , we say that $f: U \rightarrow F$ is C^r if $f|_{U^0}$ is C^r and the maps $d^i(f|_{U^0})$ admit a continuous extension $d^i f: U \times E^i \rightarrow F$ for each i as before.

1.2 In [12], one finds the concept of a C^r -manifold M *with rough boundary*, modelled on a locally convex space E . In contrast to an ordinary manifold, the charts $\phi: U \rightarrow V$ of such M are C^r -diffeomorphisms from an open subset U of M to a locally convex subset $V \subseteq E$ with dense interior. Manifolds with C^r -boundary or corners are obtained as special cases.

We shall also need $C^{r,s}$ -maps, the theory of which was developed in [1] and [2]:

1.3 Let E_1 , E_2 and F be locally convex spaces, $r, s \in \mathbb{N}_0 \cup \{\infty\}$ and $U \subseteq E_1$, $V \subseteq E_2$ be subsets. If U and V are open, then a map $f: U \times V \rightarrow F$ is called $C^{r,s}$ if the iterated directional derivatives

$$d^{i,j} f(x, y, v_1, \dots, v_i, w_1, \dots, w_j) := (D_{(v_i,0)} \cdots D_{(v_1,0)} D_{(0,w_j)} \cdots D_{(0,w_1)} f)(x, y)$$

exist for all $i, j \in \mathbb{N}_0$ with $i \leq r$, $j \leq s$ and $(x, y) \in U \times V$, $v_1, \dots, v_i \in E_1$, $w_1, \dots, w_j \in E_2$, and the map

$$d^{i,j} f: U \times V \times E_1^i \times E_2^j \rightarrow F \tag{1}$$

so obtained is continuous. If U and V are merely locally convex subsets with dense interior, we say that f is $C^{r,s}$ if $f|_{U^0 \times V^0}$ is $C^{r,s}$ and each of the maps $d^{i,j}(f|_{U^0 \times V^0})$ has a continuous extension to a mapping as in (1).

1.4 We recall from [2]:

- (a) A mapping to a (finite or infinite) product of locally convex spaces is $C^{r,s}$ if and only if all of its components are $C^{r,s}$.
- (b) If g is a C^{r+s} -map and f a $C^{r,s}$ -map taking its values in the domain of g , then also $g \circ f$ is $C^{r,s}$.
- (c) If f a $C^{r,s}$ -map, g_1 a C^r -map and g_2 a C^s -map such that $g_1 \times g_2$ takes its values in the domain of f , then $f \circ (g_1 \times g_2)$ is $C^{r,s}$.
- (d) If E_1 , E_2 and F are locally convex spaces, $U \subseteq E_1$ a locally convex subset with dense interior and $f: U \times E_2 \rightarrow F$ a $C^{r,0}$ -map such that $f(x, \cdot): E_2 \rightarrow F$ is linear for each $x \in U$, then f is $C^{r,\infty}$.
- (e) If E_1 , E_2 and F are locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ locally convex subsets with dense interior and $f: U \times V \rightarrow F$ a C^r -map, then f is $C^{r,0}$ and $C^{0,r}$.
- (f) If E_1 , E_2 and F are locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ locally convex subsets with dense interior and $f: U \times V \rightarrow F$ is $C^{r,s}$, then $f^\vee(x) := f(x, \cdot): V \rightarrow F$ is a C^s -map for each $x \in U$, and the map $f^\vee: U \rightarrow C^s(V, F)$ is C^r .

The following is obvious:

- (g) If E_1 , E_2 and F are locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ locally convex subsets with dense interior and $g: E_2 \supseteq V \rightarrow F$ is C^s , then $f: U \times V \rightarrow F$, $f(x, y) := g(y)$ is $C^{\infty,s}$.

1.5 If M_1 , M_2 and N are smooth manifolds modelled on locally convex spaces (possibly with rough boundary) and $r, s \in \mathbb{N}_0 \cup \{\infty\}$, then a map $f: M_1 \times M_2 \rightarrow N$ is called $C^{r,s}$ if it is continuous and

$$\phi \circ f \circ (\phi_1 \times \phi_2)^{-1} \text{ is } C^{r,s} \quad (2)$$

for all charts $\phi_1: U_1 \rightarrow V_1$, $\phi_2: U_2 \rightarrow V_2$ and $\phi: U \rightarrow V$ of M_1 , M_2 and N , respectively, such that

$$f(U_1 \times U_2) \subseteq U. \quad (3)$$

1.6 In view of 1.4 (b) and (c), f is $C^{r,s}$ if and only if for all $(p_1, p_2) \in M_1 \times M_2$, there exist ϕ_1, ϕ_2 and ϕ with (2) and (3) such that $p_1 \in U_1$ and $p_2 \in U_2$.

1.7 If $f: M_1 \times M_2 \rightarrow N$ (as in 1.5) is $C^{r,s}$, then $f^\vee(p_1) := f(p_1, \cdot): M_2 \rightarrow N$ is C^s for each $p_1 \in M_1$.

To see this, let $p_2 \in M_2$ and pick charts ϕ_1, ϕ_2 and ϕ with $p_1 \in U_1$ and $p_2 \in U_2$ as in 1.6. Thus $g := \phi \circ f \circ (\phi_1 \times \phi_2)^{-1}: V_1 \times V_2 \rightarrow V$ is a $C^{r,s}$ -map. Then $g^\vee(\phi_1(p_1)): V_2 \rightarrow V$ is C^s , by 1.4 (f). As a consequence, also $f^\vee(p_1)|_{U_2} = \phi^{-1} \circ g^\vee(\phi_1(p_1)) \circ \phi_2$ is C^s . Since p_2 was arbitrary, it follows that $f^\vee(p_1)$ is C^s .

The following fact is well known and easy to check.

1.8 If G is a C^k -regular Lie group and $\gamma \in C^k([0, 1], L(G))$, then

$$\text{Evol}(\gamma)(s) = \text{evol}(s\gamma_s)$$

for each $s \in [0, 1]$, where $\gamma_s: [0, 1] \rightarrow L(G)$, $\gamma_s(t) := \gamma(st)$.

1.9 If G is a Lie group and $r \in \mathbb{N}_0 \cup \{\infty\}$, then also $C^r([0, 1], G)$ is a Lie group, as is well known. If $\phi: U \rightarrow V \subseteq E$ is a chart for G such that $e \in U$ and $\phi(e) = 0$, then $C^r([0, 1], U)$ is open in $C^r([0, 1], G)$, the set $C^r([0, 1], V)$ is open in $C^r([0, 1], E)$, and the map $\phi_*: C^r([0, 1], U) \rightarrow C^r([0, 1], V)$, $\gamma \mapsto \phi \circ \gamma$ is a C^∞ -diffeomorphism (see, e.g., [12]; cf. [7]).

1.10 Note that $C_*^r([0, 1], E) := \{\gamma \in C_*^r([0, 1], E): \gamma(0) = 0\}$ is a closed vector subspace of $C_*^r([0, 1], E)$ in the preceding situation. Since ϕ_* takes $C^r([0, 1], U) \cap C_*^r([0, 1], G)$ onto $C^r([0, 1], V) \cap C_*^r([0, 1], E)$, we see that $C_*^r([0, 1], G)$ is a Lie subgroup of $C^r([0, 1], G)$ modelled on the closed vector subspace $C_*^r([0, 1], E)$. This implies that a map to $C_*^r([0, 1], G)$ is smooth if and only if it is smooth as a map to $C^r([0, 1], G)$ (cf. [3, Lemma 10.1]).

1.11 If $H \subseteq G$ is a Lie subgroup modelled on a closed vector subspace, then so is $C^r([0, 1], H) \subseteq C^r([0, 1], G)$ (entailing that a map to $C^r([0, 1], H)$ is smooth if and only if it is smooth as a map to $C^r([0, 1], G)$).

In fact, the chart ϕ in 1.9 can be chosen such that $\phi(H \cap U) = F \cap V$ for some closed vector subspace $F \subseteq E$. Since ϕ_* takes $C^r([0, 1], U) \cap C^r([0, 1], H)$ onto $C^r([0, 1], V) \cap C^r([0, 1], F)$, we see that $C^r([0, 1], H)$ is a Lie subgroup modelled on the closed vector subspace $C_*^r([0, 1], F)$.

1.12 Let G be a Lie group and $\mu: G \times G \rightarrow G$ be the group multiplication. Then TG is a Lie group with respect to the multiplication

$$T\mu: T(G \times G) \rightarrow TG,$$

identifying the left hand side with $TG \times TG$. The zero section $\theta: G \rightarrow TG$, $g \mapsto 0 \in T_g(G)$ is a smooth group homomorphism, and actually an isomorphism onto a Lie subgroup modelled on a closed vector subspace. Hence also $\theta_*: C^r([0, 1], G) \rightarrow C^r([0, 1], TG)$, $\gamma \mapsto \theta \circ \gamma$ is an isomorphism onto such a Lie subgroup, by 1.11. Likewise, the inclusion map $\alpha: L(G) \rightarrow TG$ is an isomorphism onto a Lie subgroup modelled on a closed vector subspace, and hence so is $\alpha_*: C^r([0, 1], L(G)) \rightarrow C^r([0, 1], TG)$, $\gamma \mapsto \alpha \circ \gamma$. This will be useful later.

1.13 If M is a manifold, E a locally convex space and $f: M \rightarrow E$ a C^1 -map, we identify TE with $E \times E$ as usual and let $df: TM \rightarrow E$ be the second component of $Tf: TM \rightarrow E \times E$. Thus $Tf(v) = (f(p), df(v))$ if $v \in T_pM$.

2 Auxiliary results

We first establish differentiability properties for some relevant maps.

Recall that the left logarithmic derivative $\delta^\ell(\gamma): [0, 1] \rightarrow L(G)$ of a C^1 -curve $\gamma: [0, 1] \rightarrow G$ in a Lie group G is defined via

$$\delta^\ell(\gamma)(t) := \gamma(t)^{-1} \cdot \gamma'(t) \quad \text{for } t \in [0, 1].$$

Lemma 2.1 *Let G be a Lie group and $k \in \mathbb{N}_0 \cup \{\infty\}$. Then the map*

$$D: C^{k+1}([0, 1], G) \rightarrow C^k([0, 1], TG), \quad \gamma \mapsto \gamma'$$

is a smooth group homomorphism, and also the following map is smooth:

$$\delta^\ell: C^{k+1}([0, 1], G) \rightarrow C^k([0, 1], L(G)), \quad \gamma \mapsto \delta^\ell(\gamma).$$

Proof. Let E be the modelling space for G . The map

$$D_E: C^{k+1}([0, 1], E) \rightarrow C^k([0, 1], E), \quad \gamma \mapsto \gamma'$$

is continuous linear (see [2]; cf. also [8, Lemma A.1 (d)]). If $\mu: G \times G \rightarrow G$ is the group multiplication in G , then the map $(\gamma, \eta) \mapsto \gamma\eta = \mu \circ (\gamma, \eta)$ is the multiplication in $C^{k+1}([0, 1], G)$. Now $(\gamma\eta)'(t) = T\mu(\gamma'(t), \eta'(t))$ for $\gamma, \eta \in C^{k+1}([0, 1], G)$ and $t \in [0, 1]$. Hence $(\gamma\eta)' = T\mu \circ (\gamma', \eta')$, which is the product of γ' and η' in $C^k([0, 1], TG)$. Thus D is a homomorphism of groups, and hence it will be smooth if we can show its smoothness on the open identity neighbourhood $\Omega := C^{k+1}([0, 1], U)$ for some chart $\phi: U \rightarrow V \subseteq E$ around e , with $\phi(e) = 0$. Then $\phi_*: C^{k+1}([0, 1], U) \rightarrow C^{k+1}([0, 1], V)$ is a chart for $C^{k+1}([0, 1], G)$ and $(T\phi)_*: C^k([0, 1], TU) \rightarrow C^k([0, 1], V \times E)$, $\gamma \mapsto (T\phi) \circ \gamma$ a chart for $C^k([0, 1], TG)$. Since

$$(T\phi)_* \circ D|_{\Omega} = D_E \circ \phi_*$$

(as $T\phi(\gamma'(t)) = (\phi \circ \gamma)'(t)$), we see that $D|_{\Omega} = ((T\phi)_*)^{-1} \circ D_E \circ \phi_*$ is smooth. Hence D is a smooth homomorphism.

Let $\Lambda: C^{k+1}([0, 1], G) \rightarrow C^k([0, 1], G)$ be the inclusion map (which is a smooth homomorphism), $\sigma: G \rightarrow G$, $g \mapsto g^{-1}$ be the inversion map and $\sigma_*: C^k([0, 1], G) \rightarrow C^k([0, 1], G)$, $\gamma \mapsto \sigma \circ \gamma = \gamma^{-1}$ be the smooth inversion map in $C^k([0, 1], G)$. Let $\theta: G \rightarrow TG$, $\alpha: L(G) \rightarrow TG$ as well as $\theta_*: C^k([0, 1], G) \rightarrow C^k([0, 1], TG)$ and $\alpha_*: C^k([0, 1], L(G)) \rightarrow C^k([0, 1], TG)$ be as in 1.12. Finally, let

$$(T\mu)_*: C^k([0, 1], TG) \times C^k([0, 1], TG) \rightarrow C^k([0, 1], TG), \quad (\gamma, \eta) \mapsto T\mu \circ (\gamma, \eta)$$

be the smooth group multiplication of $C^k([0, 1], TG)$. Then

$$\delta^\ell(\gamma) = (T\mu)_*(\theta_*(\sigma_*(\gamma)), D(\gamma))$$

for $\gamma \in C^{k+1}([0, 1], G)$ and thus

$$\alpha_* \circ \delta^\ell = (T\mu)_* \circ (\theta_* \circ \sigma_* \circ \Lambda, D),$$

with D as in Lemma 2.2. Thus $\alpha_* \circ \delta^\ell$ is smooth, i.e., δ^ℓ is smooth as a map to $C^k([0, 1], TG)$. Then δ^ℓ is also smooth as a map to $C^k([0, 1], L(G))$ (see 1.12), which completes the proof. \square

It will be useful later that the C^r -manifold structure on $C^{k+1}([0, 1], G)$ is initial with respect to suitable maps.

Lemma 2.2 *Let G be a Lie group, $k \in \mathbb{N}_0$, $r \in \mathbb{N}_0 \cup \{\infty\}$,*

$$D: C^{k+1}([0, 1], G) \rightarrow C^k([0, 1], TG), \quad \gamma \mapsto \gamma'$$

and $\iota: C^{k+1}([0, 1], G) \rightarrow C([0, 1], G)$, $\gamma \mapsto \gamma$ be the inclusion map. If M is a C^r -manifold and $f: M \rightarrow C^{k+1}([0, 1], G)$ a map such that both $\iota \circ f$ and $D \circ f$ are C^r , then f is C^r .

Proof. Step 1. Let E be the modelling space of G . Then the map

$$(\iota_E, D_E): C^{k+1}([0, 1], E) \rightarrow C([0, 1], E) \times C^k([0, 1], E), \quad \gamma \mapsto (\gamma, \gamma')$$

is linear and a topological embedding with closed image (see [2]; cf. also [8, Lemma A.1 (d)]). Therefore, a map $h: M \rightarrow C^{k+1}([0, 1], E)$ is C^r if and only if both $\iota_E \circ h$ and $D_E \circ h$ are C^r (cf. [3, Lemmas 10.1 and 10.2]).

Step 2. Let a chart $\phi: U \rightarrow V \subseteq E$ for G and the pushforward $\phi_*: C^{k+1}([0, 1], U) \rightarrow C^{k+1}([0, 1], V)$ be as in 1.9.

Each $p \in M$ has an open neighbourhood $P \subseteq M$ such that $f(q)f(p)^{-1} \in U$ for all $q \in P$. As it suffices to show that $f|_P$ is C^r , we may assume $P = M$. Consider $\eta := f(p)^{-1} \in C^{k+1}([0, 1], G)$ and the right translation mappings $\rho_\eta: C^{k+1}([0, 1], G) \rightarrow C^{k+1}([0, 1], G)$ and $R_\eta: C([0, 1], G) \rightarrow C([0, 1], G)$ given by $\gamma \mapsto \gamma\eta$ (which are smooth). Then

$$g := \rho_\eta \circ f: M \rightarrow C^{k+1}([0, 1], G)$$

is C^r as a function to $C([0, 1], G)$ (as $\iota \circ g = R_\eta \circ \iota \circ f$). Also, $D \circ g = \rho_{\eta'} \circ D \circ f$ is C^r , where we use the smooth right translation map

$$\rho_{\eta'}: C^k([0, 1], TG) \rightarrow C^k([0, 1], TG), \quad \gamma \mapsto \gamma\eta'.$$

Now $h := \phi_* \circ g: M \rightarrow C^{k+1}([0, 1], E)$ is C^r as a map to $C(M, E)$ (as $\iota_E \circ h = \tilde{\phi}_* \circ \iota \circ g$ with the C^∞ -diffeomorphism $\tilde{\phi}_*: C([0, 1], U) \rightarrow C([0, 1], V)$, $\gamma \mapsto \phi \circ \gamma$). Furthermore, $D_E \circ h = (d\phi)_* \circ D \circ g$ is C^r , using that $(d\phi)_*: C^k([0, 1], TU) \rightarrow C^k([0, 1], E)$ is smooth as it is the second component of the chart

$$(T\phi)_*: C^k([0, 1], TU) \rightarrow C^k([0, 1], TV) \cong C^k([0, 1], V) \times C^k([0, 1], E).$$

By Step 1, h is C^r . As a consequence, g is C^r and hence also f . \square

Lemma 2.3 *Let G be a Lie group, $k \in \mathbb{N}_0$, $r \in \mathbb{N}_0 \cup \{\infty\}$,*

$$\delta^\ell: C^{k+1}([0, 1], G) \rightarrow C^k([0, 1], L(G)), \quad \gamma \mapsto \delta^\ell(\gamma)$$

and $\iota: C^{k+1}([0, 1], G) \rightarrow C([0, 1], G)$ be the inclusion map. If M is a C^r -manifold and $f: M \rightarrow C^{k+1}([0, 1], G)$ a map such that both $\iota \circ f$ and $\delta^\ell \circ f$ are C^r , then f is C^r .

Proof. We show by induction on $j = 0, \dots, k$ that f is C^r as a map to $C^{j+1}([0, 1], G)$. The same notation, δ^ℓ , will be used for the corresponding map $C^{j+1}([0, 1], G) \rightarrow C^j([0, 1], L(G))$. Let μ, θ, α and the pushforwards $\theta_*: C^j([0, 1], G) \rightarrow C^j([0, 1], TG)$ and $\alpha_*: C^j([0, 1], L(G)) \rightarrow C^j([0, 1], TG)$ be as in 1.12. If $x \in M$, we have $D(f(x))(t) = f(x)(t) \cdot \delta^\ell(f(x))(t)$ and hence $D(f(x)) = T\mu \circ (\theta \circ (f(x)), \alpha \circ (\delta^\ell(f(x))))$, i.e.,

$$D \circ f = (T\mu)_* \circ (\theta_* \circ f, \alpha_* \circ \delta^\ell \circ f) \quad (4)$$

with the smooth group multiplication $(T\mu)_*: C^j([0, 1], TG) \times C^j([0, 1], TG) \rightarrow C^j([0, 1], TG)$. If $j = 0$, then $\theta_* \circ f = \theta_* \circ \iota \circ f$ is C^r , whence also $D \circ f$ is C^r (by (4)) and hence also f (by Lemma 2.2).

If $j > 0$, let us make the inductive hypothesis that f is C^r as a map to $C^j([0, 1], G)$. Then (4) shows that $D \circ f$ is C^r as a map to $C^j([0, 1], TG)$. Hence f is C^r as a map to $C^{j+1}([0, 1], G)$, by Lemma 2.2. \square

Lemma 2.4 *Let G be a Lie group, $k \in \mathbb{N}_0 \cup \{\infty\}$ and*

$$\delta^\ell: C_*^{k+1}([0, 1], G) \rightarrow C_*^k([0, 1], L(G)) \quad (5)$$

be left logarithmic differentiation. Let $\mathbf{1} \in C_^{k+1}([0, 1], G) =: P$ be the constant function $t \mapsto e$. Then, after identifying $T_{\mathbf{1}}P$ with $C_*^{k+1}([0, 1], L(G))$ in a suitable way, the tangent map of δ^ℓ at $\mathbf{1}$ becomes the map*

$$C_*^{k+1}([0, 1], L(G)) \rightarrow C^k([0, 1], L(G)), \quad \gamma \mapsto \gamma'. \quad (6)$$

Proof. Let $U \subseteq G$ be an open symmetric identity neighbourhood on which a chart $\phi: U \rightarrow V \subseteq L(G)$ is defined, with $\phi(e) = 0$ and $d\phi|_{L(G)} = \text{id}_{L(G)}$. Let $\mu: G \times G \rightarrow G$ be the multiplication of G and $\sigma: G \rightarrow G$ be inversion.

Then $D_U := \{(g, h) \in U \times U : gh \in U\}$ is an open subset of $U \times U$, and hence $D_V := (\phi \times \phi)(D_U)$ is an open subset of $V \times V$. The maps

$$\tau := \phi \circ \sigma \circ \phi^{-1} : V \rightarrow V \quad \text{and}$$

$$\nu := \phi \circ \mu \circ (\phi^{-1} \times \phi^{-1}) : D_V \rightarrow V$$

are smooth and $(V, D_V, \nu, 0)$ is a local group as in [20, Definition II.1.10], with inverses given by $x^{-1} := \tau(x)$. Consider the restriction

$$\delta_U^\ell : C_*^{k+1}([0, 1], U) \rightarrow C^k([0, 1], L(G))$$

of δ^ℓ from (5) and the map $\delta_V^\ell : C_*^{k+1}([0, 1], V) \rightarrow C^k([0, 1], L(G))$ defined via

$$\delta_V^\ell(\gamma)(s) := d\nu(\gamma(s)^{-1}, \gamma(s); 0, \gamma'(s))$$

for $\gamma \in C_*^{k+1}([0, 1], V)$ and $s \in [0, 1]$. Let $\phi_* : C_*^{k+1}([0, 1], U) \rightarrow C_*^{k+1}([0, 1], V)$, $\gamma \mapsto \phi \circ \gamma$ be the pushforward. An elementary calculation shows that

$$\delta_V^\ell \circ \phi_* = \delta_U^\ell. \quad (7)$$

Since ϕ_* is a C^∞ -diffeomorphism between open subsets of $C_*^{k+1}([0, 1], G)$ and $C^{k+1}([0, 1], L(G))$, respectively, we deduce from the smoothness of δ_U^ℓ that also δ_V^ℓ is smooth. Applying the Chain Rule to (7) now gives

$$d(\delta_V^\ell) \circ T_1(\phi_*) = d(\delta_U^\ell)|_{T_1 P}. \quad (8)$$

We claim that

$$\left. \frac{d}{dt} \right|_{t=0} \delta_V^\ell(t\gamma) = \gamma' \quad (9)$$

for each $\gamma \in C_*^{k+1}([0, 1], L(G))$, i.e., $d\delta_V^\ell(0, \gamma) = \gamma'$. Since also the map $T_1(\phi_*) : T_1 P \rightarrow \{0\} \times C_*^{k+1}([0, 1], L(G)) \cong C_*^{k+1}([0, 1], L(G))$ is an isomorphism, we deduce from (8) and (9) that $d(\delta^\ell)|_{T_1 P}$ is the map in (6), up to composition with the isomorphism $T_1(\phi_*)$.

To verify (9), we only need to show that

$$\left(\left. \frac{d}{dt} \right|_{t=0} (\delta_U^\ell(t\gamma)) \right) (s) = \gamma'(s)$$

for each $s \in [0, 1]$. The point evaluation at s being continuous linear, the left hand side coincides with

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} (\delta_U^\ell(t\gamma)(s)) &= \frac{d}{dt} \Big|_{t=0} d\nu((t\gamma(s))^{-1}, t\gamma(s); 0, t\gamma'(s)) \\ &= \underbrace{d^{(2)}\nu(0, 0; 0, 0; -\gamma(s), \gamma(s))}_{=0} + d\nu(0, 0; 0, \gamma'(s)) \\ &= \gamma'(s), \end{aligned}$$

as required. \square

Lemma 2.5 *Let G be a Lie group, M be a manifold and $f: M \times [0, 1] \rightarrow G$ be a $C^{r,s}$ -map. Then $f^\vee(p) := f(p, \cdot): [0, 1] \rightarrow G$ is C^s for each $p \in M$, and the map $f^\vee: M \rightarrow C^s([0, 1], G)$ is C^r .*

Proof. The first assertion is a special case of 1.7. To establish the second, choose a chart $\phi: U \rightarrow V$ of G around e . For $p \in M$, consider the auxiliary function

$$g: M \times [0, 1] \rightarrow G, \quad h(q, t) := f(q, t)f(p, t)^{-1}.$$

Since g is continuous and $g(\{p\} \times [0, 1]) = \{e\}$, $g^{-1}(U)$ is an open superset of the compact set $\{p\} \times [0, 1]$. By the Wallace Lemma [13, 5.12], there is an open neighbourhood $P \subseteq M$ such that $P \times [0, 1] \subseteq g^{-1}(U)$ and thus $g(P \times [0, 1]) \subseteq U$. After shrinking P , we may assume that there exists a chart $\psi: P \rightarrow Q$ for M . Without loss of generality $M = P$. Let $\mu: G \times G \rightarrow G$ be the group multiplication and $\sigma: G \rightarrow G$ be inversion. Since $f(p, \cdot): [0, 1] \rightarrow G$ is C^s by 1.7, using 1.4 (g) we deduce that the map

$$\zeta: M \times [0, 1] \rightarrow G, \quad (q, t) \mapsto f(p, t)^{-1} = \sigma(f(p, t))$$

is $C^{r,s}$. Hence $g = \mu \circ (f, \zeta)$ is $C^{r,s}$, as a consequence of 1.4 (c). Thus $h := \phi \circ g \circ (\psi^{-1} \times \text{id}_{[0,1]})$ is $C^{r,s}$. Now 1.4 (f) shows that $h^\vee: Q \rightarrow C^s([0, 1], V)$ is C^r . Hence also $g^\vee = (\phi_*)^{-1} \circ h^\vee \circ \psi$ is C^r , using that the map $\phi_*: C^s([0, 1], U) \rightarrow C^s([0, 1], V)$, $\gamma \mapsto \phi \circ \gamma$ is a chart for $C^s([0, 1], G)$ and hence a C^∞ -diffeomorphism. \square

Lemma 2.6 *Let E be a locally convex space and $k \in \mathbb{N}_0 \cup \{\infty\}$. Then $S: C^k([0, 1], E) \times [0, 1] \rightarrow C^k([0, 1], E)$,*

$$S(\gamma, s)(t) := s\gamma(st) \quad \text{for } \gamma \in C^k([0, 1], E), \quad s, t \in [0, 1]$$

is a $C^{\infty,0}$ -map.

Proof. Consider the map

$$h: C^k([0, 1], E) \times [0, 1] \rightarrow C^k([0, 1], E), \quad (\gamma, s) \mapsto \gamma_s$$

(with $\gamma_s(t) := \gamma(st)$). Then

$$S(\gamma, s) = \beta(s, h(\gamma, s)), \quad (10)$$

using the scalar multiplication $\beta: \mathbb{R} \times C^k([0, 1], E) \rightarrow C^k([0, 1], E)$, $\beta(r, \gamma) := r\gamma$. Now β is continuous bilinear and hence smooth. Moreover, the map $\pi: C^k([0, 1], E) \times [0, 1] \rightarrow \mathbb{R}$, $(\gamma, s) \mapsto s$ is C^∞ and hence $C^{\infty,0}$. We claim that also h is $C^{\infty,0}$. If this is true, then (π, h) is $C^{\infty,0}$ by 1.4 (a). Since $S = \beta \circ (\pi, h)$ by (10), 1.4 (b) shows that S is $C^{\infty,0}$.

Because h is linear in its first argument, 1.4 (c) applies. Thus, the claim will hold if we can show that h is $C^{0,0}$ (i.e., C^0). To this end, consider the map

$$h^\wedge: (C^k([0, 1], E) \times [0, 1]) \times [0, 1] \rightarrow E, \quad h^\wedge(\gamma, s, t) := \gamma(st).$$

Thus $h^\wedge(\gamma, s, t) = \varepsilon(\gamma, st)$, where $\varepsilon: C^k([0, 1], E) \times [0, 1] \rightarrow E$, $\varepsilon(\gamma, s) := \gamma(s)$ is C^k (see [12], cf. [8, Proposition 11.1]) and hence $C^{0,k}$. Therefore $h = (h^\wedge)^\vee$ is continuous (by 1.4 (e)), as required. \square

3 Proof of Theorem A

Step 1. By Lemma 2.5, $\text{Evol}: C^k([0, 1], L(G)) \rightarrow C^0([0, 1], G)$ will be smooth if we can show that the map

$$\text{Evol}^\wedge: C^k([0, 1], L(G)) \times [0, 1] \rightarrow G, \quad \text{Evol}^\wedge(\gamma, s) := \text{Evol}(\gamma)(s)$$

is $C^{\infty,0}$ (using that $\text{Evol} = (\text{Evol}^\wedge)^\vee$). However, using the $C^{\infty,0}$ -map

$$S: C^k([0, 1], L(G)) \times [0, 1] \rightarrow C^k([0, 1], L(G)), \quad (\gamma, s) \mapsto s\gamma_s$$

from Lemma 2.6, 1.8 enables us to write

$$\text{Evol}^\wedge(\gamma, s) = \text{Evol}(\gamma)(s) = \text{evol}(S(\gamma, s)).$$

Thus $\text{Evol}^\wedge = \text{evol} \circ S$, which is $C^{\infty,0}$ by 1.4 (b) (being a composition of a C^∞ -map and a $C^{\infty,0}$ -map).

Step 2. Consider the inclusion map $\iota: C^{k+1}([0, 1], G) \rightarrow C^0([0, 1], G)$ and the map $\delta^\ell: C^{k+1}([0, 1], G) \rightarrow C^k([0, 1], L(G))$. Then $\iota \circ \text{Evol}$ is smooth (by Step 1) and $\delta^\ell \circ \text{Evol}$ is the identity map $C^k([0, 1], L(G)) \rightarrow C^k([0, 1], L(G))$ and hence smooth as well. Therefore $\text{Evol}: C^k([0, 1], L(G)) \rightarrow C^{k+1}([0, 1], G)$ is smooth, by Lemma 2.3.

Step 3. By 1.10, Evol is also smooth when regarded as a map

$$\text{Evol}: C^k([0, 1], L(G)) \rightarrow C_*^{k+1}([0, 1], G). \quad (11)$$

Also the map $\delta^\ell: C^{k+1}([0, 1], G) \rightarrow C^k([0, 1], L(G))$ is smooth (by Lemma 2.1), and so is its restriction

$$\delta^\ell: C_*^{k+1}([0, 1], G) \rightarrow C^k([0, 1], L(G)). \quad (12)$$

Because the mappings in (11) and (12) are mutually inverse, we conclude that they are C^∞ -diffeomorphisms. The proof is complete. \square

4 Proof of Theorem B

Consider the algebra $P = \mathbb{R}[X]$ of polynomial functions on $[0, 1]$ and its multiplicative subset $S := \{s \in P: s([0, 1]) \subseteq \mathbb{R}^\times\}$. Then

$$A := PS^{-1} = \{p/s: p \in P, s \in S\}$$

is a unital subalgebra of the Banach algebra $C[0, 1]$ of continuous real-valued functions on $[0, 1]$ (and we endow A with the induced topology). As shown in [6, §6], A has an open group A^\times of invertible elements and A^\times is a smooth (and even analytic) Lie group. Note that $H := \{a \in A: a(0) = 0\}$ is a closed non-unital subalgebra of A . Now consider the subgroup

$$K := \{a \in A^\times: a(0) = 1\}$$

of A^\times . The map $\phi: A^\times \rightarrow A^\times - \{1\}$, $a \mapsto a - 1$ is a global chart for the manifold A^\times , which takes K onto $(A^\times - 1) \cap H$. Hence K is a submanifold of A^\times modelled on H and hence a Lie (sub)group. We let G be the 1-connected covering Lie group of the connected component K_1 of 1 in K , and $q: G \rightarrow K_1$ be the universal covering homomorphism. We identify $L(G)$ with $L(K_1)$ by means of $L(q)$, and $L(K_1)$ with H by means of the isomorphism

$d\phi|_{L(K_1)}$. We now verify that G and H provide counterexamples for all of the statements (a)–(d) discussed in Theorem B.

First, we show that K_1 has the property described at the end of Theorem B. To this end, let $\gamma: (\mathbb{R}, +) \rightarrow K_1$ be a homomorphism of groups. Since K_1 is a subgroup of A^\times , [6, Proposition 6.1] shows that $\text{im}(\gamma) \subseteq \mathbb{R}^\times 1$. Then $\text{im}(\gamma) \subseteq (\mathbb{R}^\times 1) \cap K = \{1\}$, whence $\gamma = 1$ indeed.

(d) Let $\gamma: (\mathbb{R}, +) \rightarrow G$ be a homomorphism of groups. Then $q \circ \gamma = 1$, as just observed. Hence $\text{im}(\gamma)$ is a subgroup of $\ker(q)$ and hence discrete. If we assume that γ is continuous, then $\text{im}(\gamma)$ is connected. Being also discrete, it must be $\{e\}$. Hence $\gamma = e$.

(c) If G was isomorphic to E/Γ as a Lie group with a locally convex space E and discrete subgroup $\Gamma \subseteq E$, then E (like G) would be infinite-dimensional. Hence a non-zero vector $v \in E$ exists, and provides a non-constant smooth homomorphism $\mathbb{R} \rightarrow E$, $t \mapsto tv$, corresponding to a non-constant smooth homomorphism $\mathbb{R} \rightarrow G$. But we have just seen that G does not admit non-trivial continuous one-parameter groups, contradiction.

(b) The Lie algebras of both $(H, +)$ and G can be identified with the locally convex space H (with the zero Lie bracket). Both G and H are 1-connected, but they cannot be isomorphic because H has many non-constant smooth one-parameter groups (of the form $t \mapsto tv$ with non-zero $v \in H$) while G has none.

(a) The Lie algebra homomorphism $\text{id}_H: H \rightarrow H$ cannot integrate to a smooth homomorphism $\Phi: (H, +) \rightarrow G$, because for non-zero $v \in H$ the map $\gamma_v: \mathbb{R} \rightarrow G$, $t \mapsto \Phi(tv)$ then would be a smooth homomorphism with $\gamma'_v(0) = L(\Phi)(v) = v \neq 0$ and thus $\gamma_v \neq e$, contradicting the above. This completes the proof of Theorem B. \square

Remark 4.1 The author does not know whether K_1 in Theorem B can be chosen connected and simply connected. In particular, he was not able to show that K_1 can be replaced by the Lie group G just discussed.

5 Proof of Theorem C

(a) Since $\delta^\ell: C_*^1([0, 1], G) \rightarrow C([0, 1], L(G))$ is a C^∞ -diffeomorphism with inverse $\text{Evol}: C([0, 1], L(G)) \rightarrow C_*^1([0, 1], G)$, the differential

$$d(\delta^\ell)|_{T_1P}: T_1P \rightarrow C([0, 1], L(G))$$

(which coincides with $d(\delta_U^\ell)|_{T_1P}$) is an isomorphism. Since also $T_1(\phi_*)$ is an isomorphism, we deduce from (8) and (9) (applied with $k = 0$) that the map

$$D: C_*^1([0, 1], L(G)) \rightarrow C([0, 1], L(G)), \quad \gamma \mapsto \gamma'$$

is an isomorphism and hence surjective. Thus $L(G)$ is integral complete.

(b) If $(E, +)$ is C^0 -regular, then E is integral complete, by (a). Conversely, suppose that E is integral complete. Then the weak integral $\eta(t) := \int_0^t \gamma(s) ds$ exists in E for each $\gamma \in C([0, 1], E)$ and $t \in [0, 1]$, and $\eta: [0, 1] \rightarrow E$ so obtained is a C^1 -map such that $\eta(0) = 0$ and $\delta^\ell \eta = \eta' = \gamma$ (by the Fundamental Theorem of Calculus, [12]) and thus $\eta = \text{Evol}(\gamma)$. Now $\text{evol}: C([0, 1], E) \rightarrow E$, $\gamma \mapsto \int_0^1 \gamma(s) ds$ is linear, and its continuity (and hence smoothness) follows from the estimate $p(\int_0^1 \gamma(s) ds) \leq \sup\{p(\gamma(s)): s \in [0, 1]\}$, valid for each continuous seminorm p on E and $\gamma \in C([0, 1], E)$.

(c) If $(E, +)$ is C^1 -regular, it is regular and thus E is Mackey complete, by [20, Remark II.5.3 (b)]. Conversely, suppose that E is Mackey complete. Then the weak integral $\eta(t) := \int_0^t \gamma(s) ds$ exists in E for each $\gamma \in C^1([0, 1], E)$ and $t \in [0, 1]$, and we conclude as in the proof of (b) that $(E, +)$ is C^1 -regular.

(d) According to [23, p. 267], the Example 4.6.110 on p. 244 in [25] furnishes a locally convex space E which is Mackey complete but does not have the metric convex compactness property. Thus E fails to be integral complete, and hence $(E, +)$ is a Lie group which is C^1 -regular (by (c)) but not C^0 -regular (by (b)). \square

References

- [1] Alzaareer, H., *Lie groups of mappings on non-compact spaces and manifolds*, Ph.D.-thesis in preparation.
- [2] Alzaareer, H. and A. Schmeding, *Differentiable mappings on products with different degrees of differentiability in the two factors*, manuscript in preparation, Universität Paderborn.

- [3] Bertram, W., H. Glöckner and K.-H. Neeb, *Differential calculus over general base fields and rings*, Expo. Math. **22** (2004), 213–282.
- [4] Dahmen, R., *Direct limit constructions in infinite dimensional Lie theory*, Ph.D.-thesis, Universität Paderborn; <http://nbn-resolving.de/urn:nbn:de:hbz:466:2-239>
- [5] Glöckner, H., *Lie groups without completeness restrictions*, Banach Center Publ. **55** (2002), 43–59.
- [6] Glöckner, H., *Algebras whose groups of units are Lie groups*, Studia Math. **153** (2002), 147–177.
- [7] Glöckner, H., *Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups*, J. Funct. Anal. **194** (2002), 347–409.
- [8] Glöckner, H., *Lie groups over non-discrete topological fields*, preprint, [arXiv:math/0408008v1](https://arxiv.org/abs/math/0408008v1).
- [9] Glöckner, H., *Fundamentals of direct limit Lie theory*, Compositio Math. **141** (2005), 1551–1577.
- [10] Glöckner, H., *Direct limits of infinite-dimensional Lie groups*, pp. 243–280 in: K.-H. Neeb and A. Pianzola (eds), “Developments and Trends in Infinite-Dimensional Lie Theory,” Progr. Math. **288**, Birkhäuser, Boston, 2011.
- [11] Glöckner, H., *Regularity in Milnor’s sense for direct limits of infinite-dimensional Lie groups*, in preparation.
- [12] Glöckner, H. and K.-H. Neeb, “Infinite-Dimensional Lie Groups,” Vol. 1, book in preparation.
- [13] Kelley, L., “General Topology,” Springer, New York, 1975.
- [14] Kriegl, A. and P. W. Michor, “The Convenient Setting of Global Analysis,” AMS, Providence, 1997.
- [15] Kriegl, A. and P. W. Michor, *Regular infinite-dimensional Lie groups*, J. Lie Theory **7** (1997), 61–99.
- [16] Michor, P. W., “Manifolds of Differentiable Mappings,” Shiva Publishing, Orpington, 1980.
- [17] Michor, P. W. and J. Teichmann, *Description of infinite-dimensional abelian regular Lie groups*, J. Lie Theory **9** (1999), 487–489.

- [18] Milnor, J., *Remarks on infinite-dimensional Lie groups*, pp.1007–1057 in: B. S. DeWitt and R. Stora (eds.), “Relativité, groupes et topologie II,” North-Holland, Amsterdam, 1984.
- [19] Neeb, K.-H., *Central extensions of infinite-dimensional Lie groups*, Ann. Inst. Fourier (Grenoble) **52** (2002), 1365–1442.
- [20] Neeb, K.-H., *Towards a Lie theory of locally convex groups*, Jpn. J. Math. **1** (2006), 291–468.
- [21] Neeb, K.-H. and F. Wagemann, *Lie group structures on groups of smooth and holomorphic maps on non-compact manifolds*, Geom. Dedicata **134** (2008), 17–60.
- [22] Omori, H., Y. Maeda, A. Yoshioka, and O. Kobayashi, *On regular Fréchet-Lie groups*, IV, Tokyo J. Math. **5** (1982), 365–398.
- [23] Voigt, J., *On the convex compactness property for the strong operator topology*, Note Mat., **12** (1992), 259–269.
- [24] von Weizsäcker, H., *In which spaces every curve is Lebesgue-Pettis-integrable?*, preprint, [arXiv:1207.6034v1](https://arxiv.org/abs/1207.6034v1).
- [25] Wilansky, A., “Modern Methods in Topological Vector Spaces,” McGraw–Hill, 1978.

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